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Compressible Two-Dimensional Solid Jets in Proximity to the Ground

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Introduction

A SOLUTION of the compressible inviscid fluid-dynamic equations describing the fluid-dynamic field of a jet starting from a nozzle and impinging on a flat surface is presented. The analysis is made in the odograph plane in which the motion equation is linear and enables us to write the solution of the problem as a sum of elementary separable solutions of the equation. Because of the discontinuous boundary conditions, the dominion of integration is divided into two different zones, and in each of these the stream function is expressed as a Fourier series. The calculation of quantities of interest enabled us to evaluate the influence of the compressibility, of the order of magnitude of 20% for high subsonic Mach numbers.

Contents

A solid jet is by definition a jet which upon impingement exhibits a line of stagnation points. The understanding of the impingement features is important for VTOL research (ejectors, helicopter rotors, and jet engines). The solution of this problem for the incompressible case was attained in 1962 by Strand[1,2] by means of a conformal mapping technique which cannot be extended to the compressible regime.

Our procedure, an extension of previously published work,[3] starts from the equation of motion written in terms of the odograph independent variables; V, modulus of the velocity vector V; and θ, angle between V and a reference axis. In this formulation the equations are linear and the solution can be represented as a sum of products of functions of V and θ only. We easily find that the functions of θ are sinusoidal functions of a suitable argument both in the incompressible and compressible cases. The functions of V are very simple in the incompressible case, but in the compressible regime they have an expression in terms of hypergeometric functions. We will numerically calculate each of them more easily by solving a difference system of algebraic equations originated by our differential equation.

We study the fluid-dynamic field of the two-dimensional solid jet in proximity to the ground by assuming[1] the angle θ of the velocity with the x axis vanishing at $x = 0$ (Fig. 1). The equation that determines the compressible stream function [$\psi_y = \rho^+ u$, $\psi_x = -\rho v$, where u and v are the velocity components in a Cartesian system of coordinates (x,y)] written in terms of V and θ is

$$V^2\left(1-\frac{\gamma-1}{2}\frac{V^2}{a_s^2}\right)\psi_{vv}+V\left(1-\frac{\gamma-3}{2}\frac{V^2}{a_s^2}\right)\psi_v+\left(1-\frac{\gamma+1}{2}\frac{V^2}{a_s^2}\right)\psi_{\theta\theta}=0 \qquad (1)$$

where a_s is the stagnation sound velocity, γ is the specific heat coefficients ratio, and the superscript + indicates nondimensional quantities referred to the stagnation ones. By indicating with a subscript 0 the conditions at infinity, let us take $\xi = V/V_0$.

The boundary conditions associated with Eq. (1) can be written as

$$\psi = 0 \qquad 0 \le \xi \le 1; \quad \theta = \pi/2 \quad [DA_0'] \qquad (2)$$

$$\psi = 0 \qquad 0 \le \xi \le \xi_i; \quad \theta = 0 \quad [CD] \qquad (3)$$

$$\psi_\theta = 0 \qquad \xi_i \le \xi \le 1; \quad \theta = 0 \quad [BC] \qquad (4)$$

$$\psi = a = \rho_0^+ \qquad \xi = 1; \quad 0 \le \theta \le \pi/2 \quad [BA_0] \qquad (5)$$

Equation (4) is the one that imposes the angle θ of the jet to be constant at $x = 0$. In the (ξ,θ) plane Eqs. (2–5) represent a set of discontinuous boundary conditions, and we look for two different solutions for the stream function ψ.

The first expression for ψ, ψ_I, is taken to be valid for $0 \le \xi \le \xi_i$ and $0 \le \theta \le \pi/2$, whereas the second one, ψ_{II}, is valid for $\xi_i \le \xi \le 1$ and $0 \le \theta \le \pi/2$. Here, ξ_i represents the nondimensional velocity in the central point C of the nozzle.

In the first zone the solution of Eqs. (1–5) can be obtained by expressing ψ by means of a Fourier series in the following form:

$$\psi_I(\xi,\theta) = \sum_{n=1}^{\infty} a_n K_{In}(\xi) \sin[2n\theta] \qquad (6)$$

In the second zone can be found in the same way

$$\psi_{II}(\xi,\theta) = \sum_{n=1}^{\infty} [b_n K_{IIn}(\xi) + c_n h_{IIn}(\xi)] \sin[(2n-1)(\pi/2-\theta)] \qquad (7)$$

In Eqs. (6) and (7), the functions $K_{In}(\xi)$, $K_{IIn}(\xi)$, and $h_{IIn}(\xi)$ are solutions of the equation

$$\xi^2\left(1-\frac{\gamma-1}{2}\xi^2\frac{V_0^2}{a_s^2}\right)K_n''+\xi\left(1-\frac{\gamma-3}{2}\xi^2\frac{V_0^2}{a_s^2}\right)K_n'-c\times\left(1-\frac{\gamma+1}{2}\xi^2\frac{V_0^2}{a_s^2}\right)K_n=0 \qquad (8)$$

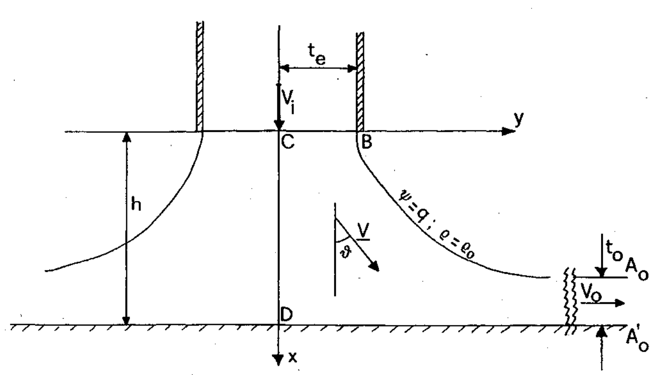

Fig. 1 Solid jet and coordinate system.

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obtained from Eq. (1) by expressing ψ as a sum of elementary functions $\psi_n = K_n(\xi)g_n(\theta)$ and using Eqs. (2–4) to determine the functions $g_n(\theta)$. The value for c is $c = 4n^2$ in the first zone and $c = (2n-1)^2$ in the second one. To avoid a singular behavior of the stream function ψ at $\xi = 0$, Eq. (8) must be solved with the condition $K_{In}(0) = 0$ in the first zone. Since Eq. (8) is of the second order, a second condition can be given: we can prescribe an arbitrary value of $K_{In}(\xi)$ at the end of the integration interval, for example, $K_{In}(\xi_i) = 1$. Equation (8) is then solved by an implicit difference method.

In the second zone, by taking $K_{IIn}(1) = 0$ and $h_{IIn}(1) = 1$ we can get from Eq. (5) a formula for the direct computation of the c_n coefficients

$$c_n = \frac{4}{\pi}\frac{a}{2n-1} \tag{9}$$

As in the first zone, we can give arbitrary conditions at the other side of the interval in order to get two different independent solutions of Eq. (8); for example, we have used $K_{IIn}(\xi_i) = 1$ and $h_{IIn}(\xi_i) = 0$.

For the complete flow solution the a_n and b_n coefficients can be determined by imposing the continuity of the stream function ψ and of its derivative with respect to ξ along the matching line $\xi = \xi_i$. In this way we get the two equations

$$\psi_I(\xi_i,\theta) = \psi_{II}(\xi_i,\theta) \tag{10}$$

$$\psi_{I\xi}(\xi_i,\theta) = \psi_{II\xi}(\xi_i,\theta) \tag{11}$$

By multiplying Eq. (10) for $\sin[(2m-1)(\pi/2-\theta)]$ and Eq. (11) for $\sin[2m\theta]$ and upon integration with respect to θ between 0 and $\pi/2$, we obtain two relations giving the b_n coefficients as a function of the a_n and vice versa, respectively. Solving by substitution these two systems of equations and taking only a finite number N of terms for the two series, we obtain a system of linear equations in the unknowns b_n only.

The convergence of the method is slow with an increasing number of coefficients anyway, because of the singularity in the boundary conditions. To avoid the use of a great number of terms in the two series we applied a numerical technique[4] to give an approximation of the neglected terms.

To obtain the physical coordinates in terms of V and θ we recall the following derivatives:

$$x_\xi = \frac{1}{\rho^+\xi}(\rho^+ \cos\theta\varphi_\xi - \sin\theta\psi_\xi)$$

$$y_\xi = \frac{1}{\rho^+\xi}(\rho^+ \sin\theta\varphi_\xi + \cos\theta\psi_\xi) \tag{12}$$

$$x_\theta = \frac{1}{\rho^+\xi}(\rho^+ \cos\theta\varphi_\theta - \sin\theta\psi_\theta)$$

$$y_\theta = \frac{1}{\rho^+\xi}(\rho^+ \sin\theta\varphi_\theta + \cos\theta\psi_\theta) \tag{13}$$

$$\varphi_\xi = (1/\rho^+)_\xi\psi_\theta - (1/\rho^+)_\theta\psi_\xi - 1/(\rho^+\xi)\psi_\theta$$

$$\varphi_\theta = \xi\psi_\xi/\rho^+ \tag{14}$$

When $\xi = \text{const}$ by integrating Eq. (13a) and taking $\xi = \xi_i$ and $\theta = \pi/2$ we have

$$\frac{h}{t_0} = \frac{\pi}{4\rho_i^+}\left\{b_1\left[K'_{II1}(\xi_i) + \frac{K_{II1}(\xi_i)}{\xi_i}\right] + c_1\left[h'_{II1}(\xi_i) + \frac{h_{II1}(\xi_i)}{\xi_i}\right]\right\} \tag{15}$$

By integration of Eq. (12b) for $\theta = \text{const}$ and then taking $\xi = 1$ and $\theta = 0$ we have

$$\frac{t_e}{t_0} = \int_{\xi_i}^{1}\frac{1}{\rho^+\xi}\sum_{n=1}^{\infty}(-1)^{n-1}[b_nK'_{IIn}(\xi_\xi) + c_nh'_{IIn}(\xi_\xi)]\,d\xi \tag{16}$$

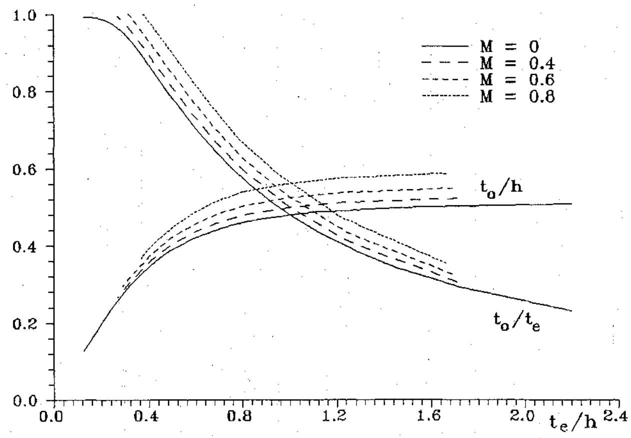

Fig. 2 Jet-contraction ratios.

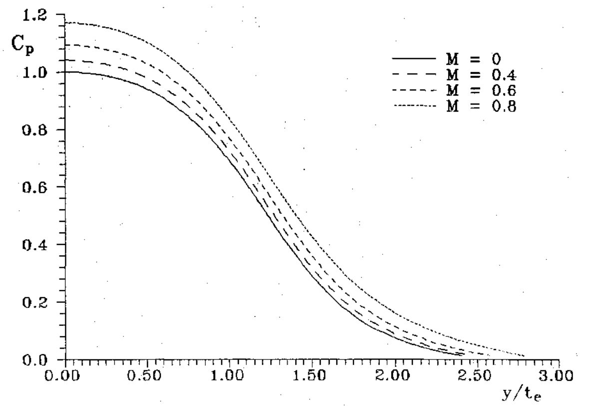

Fig. 3 Pressure coefficient along the ground, $h/2t_e = 0.5$.

Equations (15) and (16) give the ratio t_e/h as a function of ξ_i and V_0/a_s.

The pressure coefficient $C_p = (p-p_0)/(1/2\rho V_0^2)$ is given, in the incompressible and compressible case, by the relations

$$C_{p,\text{inc}} = 1 - \xi^2; \quad C_{p,\text{comp}} = 2(p^+ - p_0^+)/(\gamma\rho_0^+ V_0^2/a_s^2) \tag{17}$$

Analysis of the Results

The data of the problem can be the following: M_0 is the Mach number on the outer streamline, and $\xi_i = V_i/V_0$. Equation (15) enables us to obtain $h/t_0(x_D/t_0)$, the ratio between height from the ground and the thickness of jet at infinity; Eq. (16) gives $t_e/t_0(y_B/t_0)$, the ratio between the thickness of the jet at the exit and at infinity. Thus, we have t_e/h in terms of the data of the problem. Figure 2 shows the ratio t_0/t_e and t_0/h vs t_e/h for different Mach numbers of the outer streamline. These curves for $M_0 = 0$ practically coincide with those obtained by Strand.[1]

Figure 3 shows the pressure distribution along the ground for one of the values of $t_e/2h$ computed by Strand and the same Mach numbers previously used. As can be observed, for $M_0 = 0$ the accord with Strand's data is very good, and the magnitude of the correction for high subsonic Mach number is about 20%.

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